SOLUTIONS OF PROBLEMS IN NUMBER TWO.

Solutions of problems in number two have been received as follows:

From Prof. W. P. Casey, 250, 251, 257; George Eastwood, 255, 257; Prof. E. J. Edmunds, 251, 253, 255; Prof. A. B. Evans, 257; Prof. A. Hall, 256; A. S. Hathaway, 257; W. E. Heal, 250, 251, 252, 253, 254, 255; Prof. W. W. Johnson, 257; Chas. H. Kummell, 251, 254, 256, 257; Prof. J. H. Kershner, 250, 251, 252, 253, 254, 255, 256, 257; Artemas Martin, 254; Prof. M. C. Stevens, 250, 251; E. B. Seitz, 257; Prof. J. Scheffer, 250, 251, 253, 254, 255, 256, 257.

Besides the answer given on page 85, Query 1 was, also, answered by Prof. W. P. Casey and Prof. J. H. Kershner.

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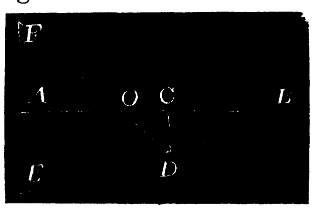
250. "Divide the line AB, geometrically, into three parts that shall be in harmonic proportion."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

As the ratio of the harmonic section of a line or angle may have any value, real or imaginary, a line or angle may be cut harmonically in an infinite number of ways; but the ratio of the harmonic section, or the position of one of the two conjugates is of course sufficient to determine the particular harmonic section in the case of a given line or angle.



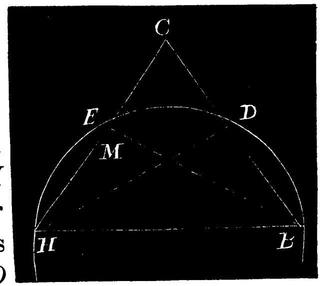
Construction. — Assume any point C in the given line, and draw CD at right angles to it. Take any point D in this line, and join BD and produce it to meet the perpendicular AE in the point E. Make $AF = AE$; join FD, intersecting AC in the point O; then is the line AB divided in the required proportion, that is; $AB : BC :: AO : OC$. OB is evidently a harmonic mean between AB and BC; and so is AC one between AB and AO.

If it be required to divide AB into three parts that shall have this proportion, and that AB shall not form one of its terms, that is, having $AO : CB :: AO - OC : OC - CB$, it is also unlimited, but may be solved in a similar manner. It may be shown from the proportion $AB:BC::AO$ OC that $BO(AB + BC) = 2AB.BC$: and also, that $OB(AO - OC) = 2AO.OC$. If $AO.CB$, $AO - CB$, $AC.OB$, $AC^2 + OB^2$, or $AC^2 - OB^2$ were given the problem would be limited, and there would be only one solution.

251. "Show that when the two lines, which bisect two angles of a triangle, are equal the triangle is isosceles."

SOLUTION BY PROF. M. C. STEVENS, LAFAYETTE, IND.

Let ABC be a triangle such that when the angles A and B are bisected by AD and BE, $AD = BE$. Then the angle A must be equal to the angle B, and the triangle be isosceles.



Proof.—Through the three points A, B and D pass the circumference of a circle. It will also pass through E. For if it meets BE in M between B and E the arc DB must be greater than MD since DAB which is equal to EAD is greater than the angle MAD. Also the arc MD is equal to the arc MA since the angle DBM is equal to the angle MBA. Whence the arc BDM is greater than DMA, and consequently the chord BM is greater than the chord AD. But $AD = BE$; $\therefore$ BM is greater than BE which is absurd. Hence the circle which passes through A, B and D cannot cut BE between B and E.

In like manner it may be shown that the circle cannot cut BE beyond E. It must therefore pass through E. Hence the angle DBE which is half of B is equal to EAD which is half of A. Therefore the angle B is equal to the angle A and the triangle is isosceles.

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252. "If the roots of a given cubic equation be not *real* and *positive* show that the equation can be transformed into another, of the same degree, in which *all* the roots are real and positive."

SOLUTION BY W. E. HEAL, WHEELING, INDIANA.

Let x_1, x_2, x_3 and x_1^2, x_2^2, x_3^2 be the respective roots of the equations

$$x^3+ax^2+bx+c=0, \tag{1}$$

$$x^3+Ax^2+Bx+C=0. \tag{2}$$

We have $x_1+x_2+x_3=-a$, $x_1x_2+x_1x_3+x_2x_3=b$, $x_1x_2x_3=-c$; $x_1^2+x_2^2+x_3^2=-A$, $x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2=B$, $x_1^2x_2^2x_3^2=-C$.

$$x_1^2+x_2^2+x_3^2=a^2-2b \qquad =-A. \qquad \therefore A=2b-a^2. \tag{3}$$

$$x_1^4+x_2^4+x_3^4=a^4-4a^2b+4ac+2b^2=A^2-2B. \quad \therefore B=b^2-2ac. \tag{4}$$

$$x_1^2x_2^2x_3^2=c^2=-C. \quad \therefore C=-c^2. \tag{5}$$

If all the roots of an equation are real we can transform it by (3), (4) and (5) into another in which the roots shall be the squares of the roots of the given equation and therefore *all* the roots will be positive whatever the signs of the roots of the given equation.

If two roots are imaginary they must be of the form $p+q\sqrt{-1}$, $p-q\sqrt{-1}$. Let r denote the real root, which can be expressed by Cardan's formula.

We have $\qquad r+(p+q\sqrt{-1})+(p-q\sqrt{-1})=a.$

$$\therefore\ p=\tfrac{1}{2}(a-r).$$

Let $x=z-\frac{1}{2}(a-r)$. The equation found by substitution must have $\frac{1}{2}(3r-a)$, $q\sqrt{-1}$, $-q\sqrt{-1}$ for its roots. Transforming by (3), (4) and (5) we get an equation whose roots are $\frac{1}{4}(3r-a)^2$, $-q^2$, q^2. Transforming again we have a cubic whose roots are $\frac{1}{16}(3r-a)^4$, q^4, q^4, all *real* and *positive*.

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253. "If $f(x)$ be a function whose roots are all real, show that the differential of the second order of that function has all its roots imaginary."

ANSWER BY PROF. KERSHNER.

The second term of the "second differential" of the general equation

$$x^n+Ax^{n-1}+B^{n-2}+\&c.=0,$$

after proper reduction, is $A(1-\frac{1}{2}n)x^{n-3}$, from an inspection of which it is evident that no change such as predicated by the proposer is even possible. The problem, therefore, as stated, involves an error.

ANSWER BY PROF. SCHEFFER.

As far as I understand this problem—and I do not notice any ambiguity of expression which might admit of different interpretation—the assertion is incorrect, as can be easily demonstrated. If, for instance, the function is of the third degree, the second differential will be of the first, which certainly cannot admit of any imaginary roots, as such occur only by pairs.

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254. "Integrate $dI=xE(e,x)dx$, where $E(e,x)$ denotes an elliptic arc, eccentricity e and abscissa x."

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURVEY, DETROIT, MICH.

Integrating by parts we have

$$I=\tfrac{1}{2}x^2E(e,x)-\tfrac{1}{2}\int_0^x x^2dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}\ \text{(semi major axis = 1)}$$

$$=\tfrac{1}{2}x^2E(e,x)-\tfrac{1}{2}I'. \tag{1}$$

Let $$R=\sqrt{[(1-x^2)(1-e^2x^2)]};\tag{2}$$

then we have
$$d(Rx)=\left(R+\frac{-(1+e^2)x^2+2e^2x^4}{R}\right)dx$$
$$=\frac{1-2(1+e^2)x^2+3e^2x^4}{R}dx$$
$$=\frac{1+(1-2e^2)x^2}{R}dx-3dI'$$
$$=\frac{1+[(1-2e^2)\div e^2]}{R}dx-\frac{1-2e^2}{e^2}d[E(e,x)]-3dI'$$
$$=\frac{1-e^2}{e^2}d[F(e,x)]-\frac{1-2e^2}{e^2}d[E(e,x)]-3dI'.\tag{3}$$

Substituting this into (1) we obtain
$$I=\tfrac{1}{2}x^2E(e,x)+\frac{1-2e^2}{6e^2}E(e,x)-\frac{-e^2}{6e^2}F(e,x)+\tfrac{1}{6}x\sqrt{[(1-x^2)(1-e^2x^2)]}.$$

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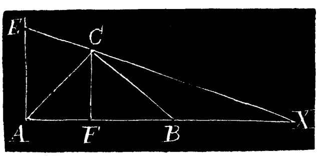
255. "In a triangle ABC the angle $A=\varphi+a$, $B=2\varphi$; supposing AB to remain fixed, while φ varies, it is required to find the rectangular equation to the locus of C, and the equations to the asymptotes.

2. With the same data as above, taking A as the origin, and AB as the axis of x; let $a=45°$, and find the envelop of a straight line which passes through C and makes an angle $4\varphi+90°$ with AX."

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Let ABC be a triangle in which $\angle A=\varphi+a$, $\angle B=2\varphi$, base $AB=\beta$, and let CD be a line which makes with AB produced an angle$=90°+4\varphi$.



Then, since $2A-B=2a$, we have
$$\tan(2A-B)=\frac{\tan 2A-\tan B}{1+\tan 2A\tan B}=\tan 2a,$$
a given quantity.

But if x, y be the coordinates of the vertex C,
$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}=\frac{2xy}{x^2-y^2},\text{ and }\tan B=\frac{y}{\beta-x}.$$

Therefore
$$\frac{\tan 2(\varphi+a)-\tan 2\varphi}{1+\tan 2(\varphi+a)\tan 2\varphi}=\frac{2y(\beta-x)x-x^2y+y^3}{x^2(\beta-x)-(\beta-3x)y^2}=\tan 2a;$$
$$\therefore\ y^3+\tan 2a(\beta-3x)y^2-(3x^2-2\beta x)y-\tan a(\beta-x)x^2=0,$$
which indicates a locus of the third degree.

2. The equation of the line DC is

$$y = x\tan(90^\circ + 4\varphi) + b, \qquad (1)$$

in which b represents the intercept AE.

Now in the triangle ACB we have $BC = \beta\sin(\varphi + \alpha) \div \sin(\varphi + \alpha + 2\varphi)$, and in the triangle BCD, we find

$$BD = \frac{-\beta\sin(\varphi - \alpha)\cos 6\varphi}{\sin[(\varphi+\alpha)+2\varphi]\cos 4\varphi},\ AD = \beta - \frac{\beta\sin(\varphi+\alpha)\cos 6\varphi}{\sin[(\varphi+\alpha)+2\varphi]\sin 4\varphi},$$

$$AE = \beta\cot 4\varphi - \frac{\beta\sin(\varphi+\alpha)\cos 6\varphi}{\sin[(\varphi+\alpha)+2\varphi]\sin 4\varphi} = b. \qquad (2)$$

Put $\tan(90^\circ + 4\varphi) = p$; then $\tan 4\varphi = -1 \div p$, $\sin 4\varphi = -1 \div (1+p^2)^{\frac{1}{2}}$, and equation (1) becomes

$$y = px - \beta p + \frac{\beta\sin(\varphi+\alpha)\cos 6\varphi\sqrt{(1+p^2)}}{\sin[(\varphi+\alpha)+2\varphi]}. \qquad (3)$$

Differentiating equation (3) with respect to the parameter p, and equating dy to zero, we obtain, by developing $\sin(\varphi+\alpha)$, $\sin[(\varphi+\alpha)+2\varphi]$ in functions of x and y, and putting $m = \sqrt{[(\beta-x)^2+y^2]}$,

$$p = \frac{\beta - x}{\sqrt{[m^2\cos^2 6\varphi - (\beta-x)^2]}}.$$

Substituting this value of p in equation (3), we obtain, after reduction,

$$[y^2 - 3(\beta-x)^2][y^2 - 3(\beta-x)^2] = 0,$$

which indicates, as the required envelop, two equal hyperbolas whose centers are at B and whose semi axes are $\sqrt{[3\beta(\beta-1)]}$ and $\sqrt{[\beta(\beta-1)]}$.

Remark.—Since this result is independent of a, it would appear that the *datum*, $a = 45^\circ$, is not necessary to the solution of the problem.

[Mr. Eastwood has not given the equation to the asymptotes. Professor Scheffer determines the equation of the asymptotes as follows: Putting $y = \mu x + \beta$ for the equation of the asymptote, he substitutes this value of y in his Eqn. of the locus, $(y^2 - x^2)[y\cos 2a + (a-x)\sin 2a] + 2xy[(a-x)\cos 2a - y\sin 2a] = 0$, and thence, by putting the coef. of x^3 and x^2, respectively $= 0$, he obtains equations (4) and (5). From (4) he derives the cubic equation, $\mu^3 - 3\mu^2\tan 2a - 3\mu + \tan 2a = 0$, from which he finds the three roots μ_1, μ_2, μ_3. Substituting the first of these roots ($\tan\frac{2}{3}a$) in (5) he finds $\beta = -\frac{1}{3}a\tan\frac{2}{3}a$. ($a = AB$.) Substituting in the eqn. $y = \mu x + \beta$, he gets $y = (x - \frac{1}{3})\tan\frac{2}{3}a$, for the eq. of one of the asymptotes.]

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256. "If the given quantities x_1, x_2, x_3, x_4 have the probable errors r_1, r_2, r_3, r_4, respectively, find the probable error r of the quantity x when $x_1 : x_2 :: x_3 + x : x_4 + x$."

SOLUTION BY PROF. HALL.

If we have given $x=\varphi(x_1, x_2, x_3, x_4)$
and r_1, r_2, r_3, r_4 are the probable errors of x_1, x_2, x_3, x_4, we have, by the theory of least squares, the probable error of x

$$=\pm\left\{\left(\frac{d\varphi}{dx_1}\right)^2 r_1^2+\left(\frac{d\varphi}{dx_2}\right)^2 r_2^2+\left(\frac{d\varphi}{dx_3}\right)^2 r_3^2+\left(\frac{d\varphi}{dx_4}\right)^2 r_4^2\right\}^{\frac{1}{2}}.$$

In the present case we have

$$x=\frac{x_1x_4-x_2x_3}{x_2-x_1}.$$

Hence we have the probable error of x

$$r=\pm\frac{1}{(x_2-x_1)^2}\sqrt{[x_2^2(x_4-x_1)^2r_1^2+x_1^2(x_3-x_4)^2r_2^2+x_2^2(x_1-x_2)^2r_3^2+x_1^2(x_2-x_1)^2r_4^2]}.$$

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257. "Within a triangle ABC, determine a point P, such that $m.PA+n.PB+r.PC$ shall be a minimum, m, n, r being constants."

SOLUTION BY PROF. W. W. JOHNSON, ANNAPOLIS, MD.

Denoting PA by ρ_1, PB by ρ_2, PC by ρ_3, we are to have

$$m\rho_1+n\rho_2+r\rho_3=\text{a minimum.}$$

When P is in the proper position the locus of

$$m\rho_1+n\rho_2=\text{a constant}$$

must touch the circle whose radius is ρ_3, otherwise by moving P along this locus we could diminish ρ_3 without increasing $m\rho_1+n\rho_2$. This locus is a Cartesian Oval; s being an arc of this curve, the cosine of the angle between ρ_1 and the tangent is evidently $d\rho_1\div ds$, and that of the angle between ρ_2 and the same end of the tangent is $-d\rho_2\div ds$; and since from the equation of the locus,

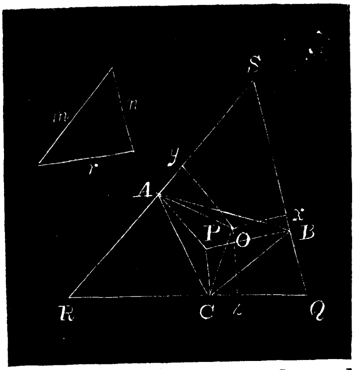
$$md\rho_1=-nd\rho_2,$$

these cosines are inversely in the ratio $m : n$. Now ρ_2 being perpendicular to the tangent, the sines of the angles at P subtended by b and a are inversely as $m : n$. A corresponding relation existing between the sines of each pair of angles, the sines of the angles subtended by a, b and c are directly as $m : n : r$.

To find P construct a triangle whose sides are m, n, r; the required angles are the supplements of the angles of this triangle, and P may be determined by the intersection of circular segments constructed on two of the sides and containing the proper angles.

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Construction.— Determine a triangle QRS having its sides pass through the vertices of the given triangle ABC and similar to that determined by the three multiples m, n, r; and having AP, BP, CP perpendicular to RS, SQ, QR respectively.



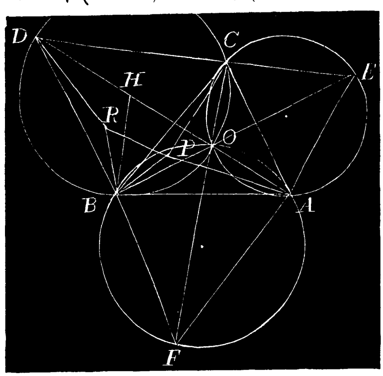
Upon AC and AB describe two segments of circles containing angles respectively = to the supplements of the angles mr, mn; the intersection of the circles will give the point P.

Join PA, PB, PC; the sides RS, SQ and QR being respectively perpendicular to PA, BP and CP, P is the point required; for, take any other point, O, and draw the perpendiculars, Oy, Ox, Oz; then, by a well known property in modern geometry, $m.PA+n.PB+r.PC=m\,.Oy+n\,.Ox+r\,.Oz$, which is less than $m.OA+n.OB+r.OC$.

When the three given multiples m, n, r are incapable of forming a triangle, this method of determining P fails, but it is easily seen, at once without any construction, that if any of the multiples m, n, r be = or > than the sum of the other two, the point itself corresponding to that multiple is that for which the sum $m.PA+n.PB+r.PC$ is the minimum.

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Dividing the expression by m, we have $OA+(n\div m)\,.\,OB+(r\div m)\,.\,OC$, which is to be a minimum.



Take any point P, within the triangle, and join PA, PB, PC. Construct the triangle BPR, such that $BP : PR : RB = m : n : r$, and construct the triangle BRD, making it similar to BPC.

Then we have $PR=(n\div m).PB$, and $RD=(r\div m).PC$; $\therefore$ $AP+PR+RD = PA+(n\div m).PB+(r\div m).PC$.

The angle $CBD=CBR+RBD=CBR+CBP=PBR$, and $BC : BD :: BP : BR$; hence the triangle BCD is similar to the triangle BPR. The point D is, therefore known.

Now $AP+PR+RD$, or $PA+(n\div m).PB+(r\div m).PC$ will be least when $APRD$ becomes a straight line. This will be the case when P is taken so that the angle APB is the supplement of BPR, and BPC or BRD is the supplement of BRP. Hence we have the following

Construction. – On the sides BC, CA, AB construct the triangles BCD, CAE, ABF, making them similar to BPR, the angle $CBD=ABF=CEA=PBR$, $BCD=ACE=AFB=BPR$, and $CAE=BAF=BDC=BRP$. Circumscribe circles about the trian's BCD, CAE, ABF.

Now since $BDC+CEA+AFB=180°$, the sum of the angles inscribed in the arcs BC, CA, AB, is $360°$; hence the circumferences intersect in one point O, which is the required point. Join OD, OE, OF, and make the angle $OBH=CBD$.

The angle BOD is equal to BCD, which by construction is equal BFA, and BFA is the supplement of AOB; hence BOD is the supplement of AOB, and AOD is, therefore, a straight line. Similarly it can be proved that BOE and COF are straight lines.

The triangle BOH being similar to BCD, and BHD to BOC, we have $OH=(n\div m).OB$, and $HD=(r\div m).OC$; $\therefore$ $AD=OA+(n\div m).OB+(r\div m).OC$, and $m.AD=n.OA+m.OB+r.OC=$ the req'd minimum.

In like manner we prove that $n.BE$ and $r.CF$ each $=$ the req'd min.

If Δ, Δ_1 are the areas of the $\triangle$ ABC and a $\triangle$ with sides m, n, r, and if α, β, γ represent the angles A, B, F, of the $\triangle$ ABF, we find,

$$OA=\frac{2m\Delta(\cot A+\cot\alpha)}{\sqrt{[2\Delta_1(a^2\cot\alpha+b^2\cot\beta+c^2\cot\gamma+4\Delta)]}},$$

$$OB=\frac{2n\Delta(\cot B+\cot\beta)}{\sqrt{[2\Delta_1(a^2\cot\alpha+b^2\cot\beta+c^2\cot\gamma+4\Delta)]}},$$

$$OC=\frac{2r\Delta(\cot C+\cot\gamma)}{\sqrt{[2\Delta_1(a^2\cot\alpha+b^2\cot\beta+c^2\cot\gamma+4\Delta)]}}.$$

PROBLEMS.

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259. By PROF. J. SCHEFFER, *Mercersberg, Pa.*—In a triangle, one side $=400$ ft., and the two adjacent angles, $70°$ and $80°$, are given; to compute the other two sides without the aid of trigonometry.

260. By GEO. M. DAY, *Lockport, N. Y.*—A sphere, radius r, rolls down the concave arc of a circle, radius R. At the beginning of the motion, the center of the sphere is on the horizontal diameter of the circle. Find the time of descent of the sphere in terms of the coordinable of its center.